Example 1

consider the function: $f(x,y) = x^2 - y^2$

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y$$

These first derivatives are zero at x* = 0 and y* = 0. The Hessian matrix of *f is:*

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\,\partial y} \\ \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

The Hessian matrix of *f* at (x*,y*) = H

The determinant H1 = 2 (positive), and the determinant H2 = 2(-2) -0(0) = -4 (negative). Then H is indefinite.

Since this matrix is neither positive definite nor negative definite, the point (x* = 0, y* = 0) is a saddle point.

Example 2

Find the critical points of the function:

$$f(x_1,x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$$

SOLUTION: The necessary conditions for the existence of an extreme point a

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 = x_1(3x_1 + 4) = 0 \quad \text{.... (1)}$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = x_2(3x_2 + 8) = 0 \quad \text{... (2)}$$

From (1) $x_1 = 0$ or $(-\frac{4}{3})$, and from (2) $x_2 = 0$ or $(-\frac{8}{3})$. Then these equations are satisfied at the points:

$(0,0)$, $(0,-\frac{8}{3})$, $(-\frac{4}{3}, 0)$, and $(-\frac{4}{3}, -\frac{8}{3})$

To find the nature of these extreme points, we have to use the sufficiency conditions. The second-order partial derivatives of *f* are given by:

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 \qquad \frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4 \qquad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8 \qquad \frac{\partial^2 f}{\partial x_1\, \partial x_2} = 0$$

The Hessian matrix of *f* is given by:

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\, \partial x_2} \\ \frac{\partial^2 f}{\partial x_2\, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

$$f(x_1,x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$$

$$\mathbf{H}(f) = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

If $J_1 = |6x_1 + 4|$ and $J_2 = \begin{vmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{vmatrix}$, the values of J_1 and J_2 and the nature of the extreme point are as given below.

Point **X**	Value of J_1	Value of J_2	Nature of **J**	Nature of **X**	$f(\mathbf{X})$
(0,0)	+4	+32	Positive definite	Relative minimum	6
$(0, -\frac{8}{3})$	+4	−32	Indefinite	Saddle point	418/27
$(-\frac{4}{3}, 0)$	−4	−32	Indefinite	Saddle point	194/27
$(-\frac{4}{3}, -\frac{8}{3})$	−4	+32	Negative definite	Relative maximum	50/3

1.7 CONSTRAINED PROBLEMS

1.7.1 Multivariable Optimization With Equality Constraints

We consider the optimization of continuous functions subjected to equality constraints:

Minimize $f = f(\mathbf{X})$

subject to

$g_j(\mathbf{X}) = 0, \qquad j = 1,2,\ldots,m$

Where: $\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$

Here m is less than or equal to n; otherwise (if $m > n$), the problem becomes overdefined and, in general, there will be no solution. There are several methods available for the solution of this problem: The Constrained variation, Jacobian method, **Methods of direct substitution**, and **Lagrange multipliers**.

1.7.1.1 Method of Direct Substitution

For a problem with *n* variables and *m* equality constraints, it is theoretically possible to solve simultaneously the *m* equality constraints and express any set of *m* variables in terms of the remaining *n* — *m* variables. When these expressions are substituted into the original objective function, there results a new objective function involving only *n* — *m* variables. The new objective function is not subjected to any constraint, and hence its optimum can be found by using the unconstrained optimization techniques.

Example 1

$$\text{Minimize:} \quad f(\mathbf{x}) = 4x_1^2 + 5x_2^2$$

$$\text{Subject to:} \quad 2x_1 + 3x_2 = 6$$

Either x_1 or x_2 can be eliminated without difficulty. Solving for x_1,

$$x_1 = \frac{6 - 3x_2}{2} \qquad \ldots\ldots\ (1)$$

Substitute for x_1 in the Objective Function, the new equivalent objective function in terms of a single variable x_2 is:

$$f(x_2) = 14x_2^2 - 36x_2 + 36$$

The constraint in the original problem has now been eliminated, and $f(x_2)$ is an unconstrained function with one independent variable.

We can now minimize the new objective function by setting the first derivative of *f* equal to zero, and solving for the optimal value of x_2:

$$f(x_2) = 14x_2^2 - 36x_2 + 36$$

$$\frac{df(x_2)}{dx_2} = 28x_2 - 36 = 0 \quad x_2^* = 1.286$$

$f''(x) = 28$ (positive), then X* is a local minimum.

Once x_2^* is obtained, then, x_1^* can be directly obtained via the relation (1): $x_1 = \frac{6 - 3x_2}{2}$, then: $x_1^* = \frac{6 - 3x_2^*}{2} = 1.071$

$$f(\mathbf{x}) = 4x_1^2 + 5x_2^2$$

$$f_{min} = 4(1.071)^2 + 5(1.286)^2 = 12.85714$$

Solving Using Excel

Problem:

$$\text{Minimize:} \quad f(\mathbf{x}) = 4x_1^2 + 5x_2^2$$

$$\text{Subject to:} \quad 2x_1 + 3x_2 = 6$$

F4 | f_x =SUMPRODUCT(B4:C4;B3:C3)

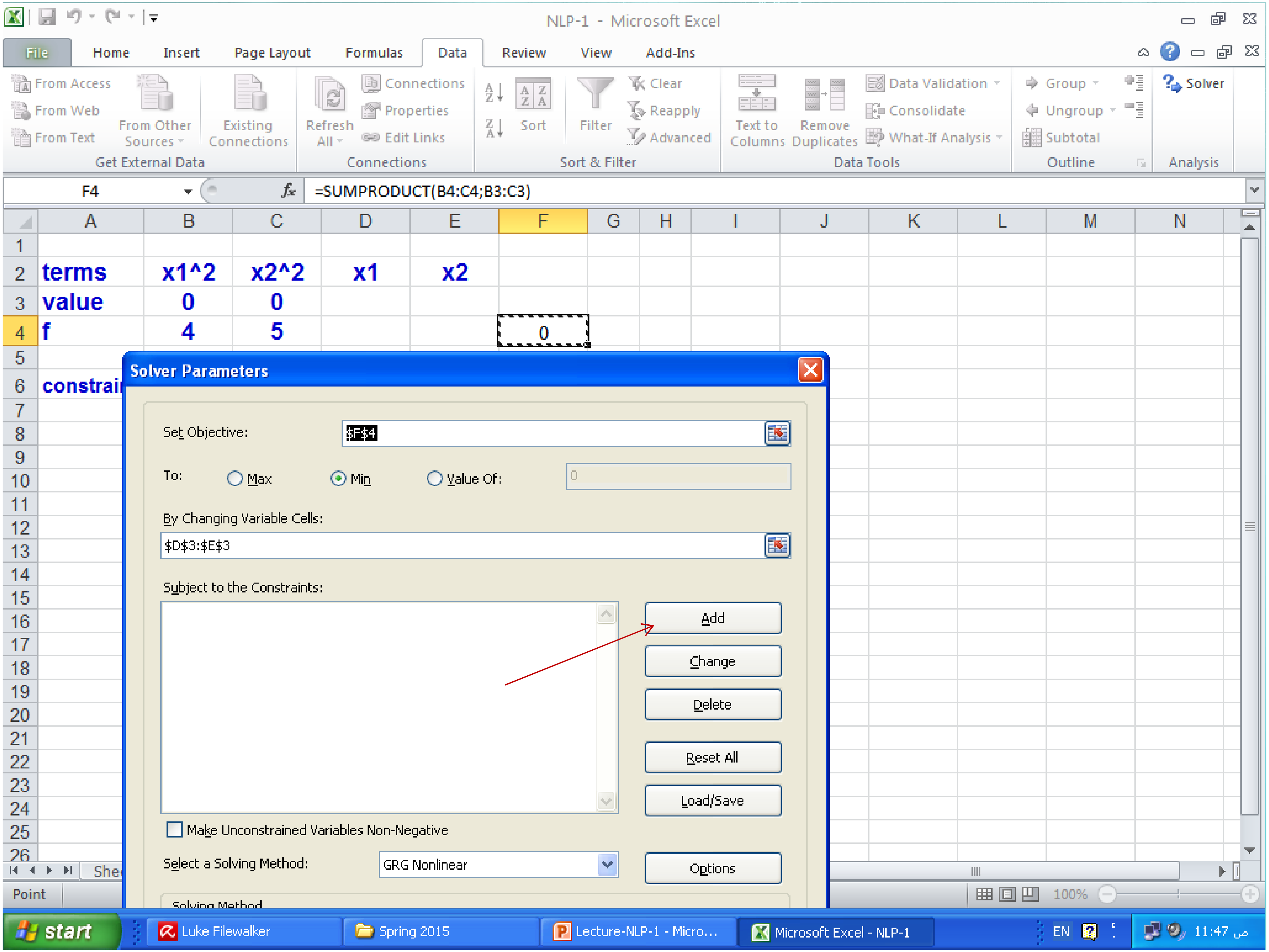
	A	B	C	D	E	F	G	H	I
1									
2	terms	x1^2	x2^2	x1	x2				
3	value	0	0						
4	f	4	5			0			
5									
6	constraint			2	3	0	=	6	
7									

B3: = D3^2

C3: = E3^2

F4: =SUMPRODUCT(B4:C4;B3:C3)

F6: = SUMPRODUCT(D6:E6;D3:E3)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x2^2
x1
x2
value
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f
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Solver Parameters
Set Objective:
$F$4
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$D$3:$E$3
Subject to the Constraints:
Add
Change
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Reset All
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Make Unconstrained Variables Non-Negative
Select a Solving Method:
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Luke Filewalker
Spring 2015
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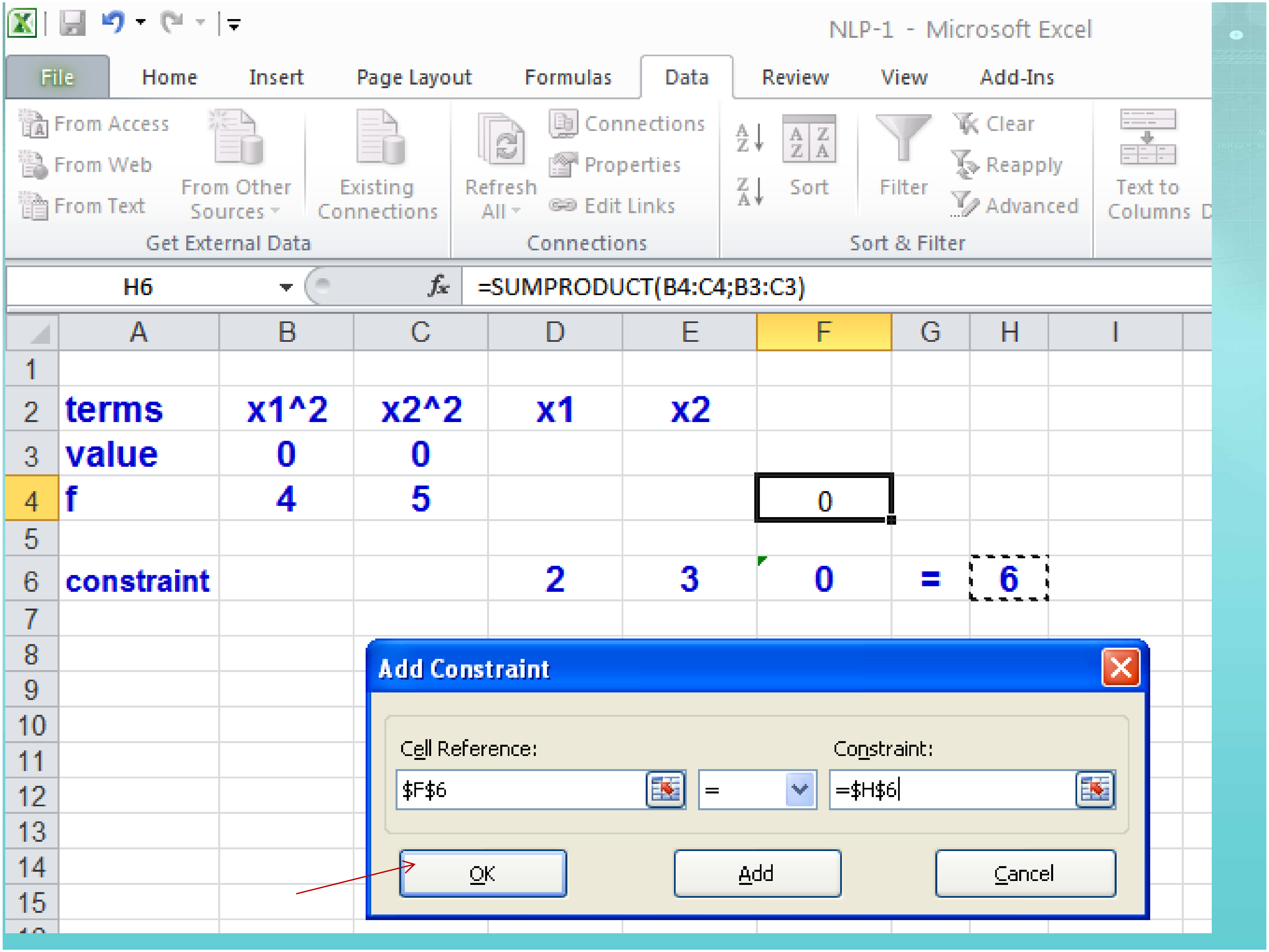
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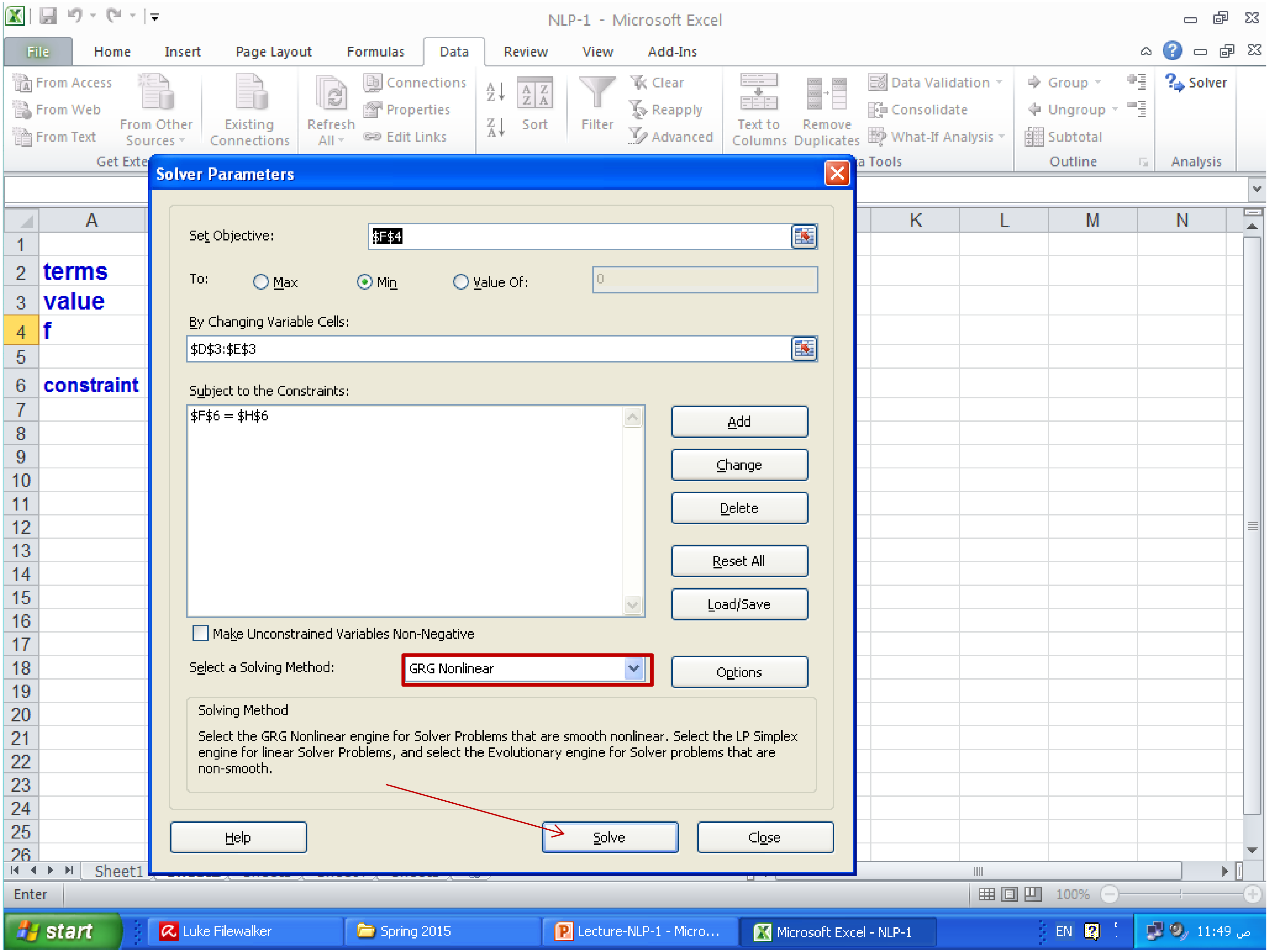
Text to Columns
H6
=SUMPRODUCT(B4:C4;B3:C3)
terms
x1^2
x2^2
x1
x2
value
0
0
f
4
5
0
constraint
2
3
0
=
6
Add Constraint
Cell Reference:
$F$6
=
Constraint:
=$H$6
OK
Add
Cancel



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Subtotal
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Analysis
terms
value
f
constraint
Solver Parameters
Set Objective:
$F$4
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$D$3:$E$3
Subject to the Constraints:
$F$6 = $H$6
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
GRG Nonlinear
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
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Luke Filewalker
Spring 2015
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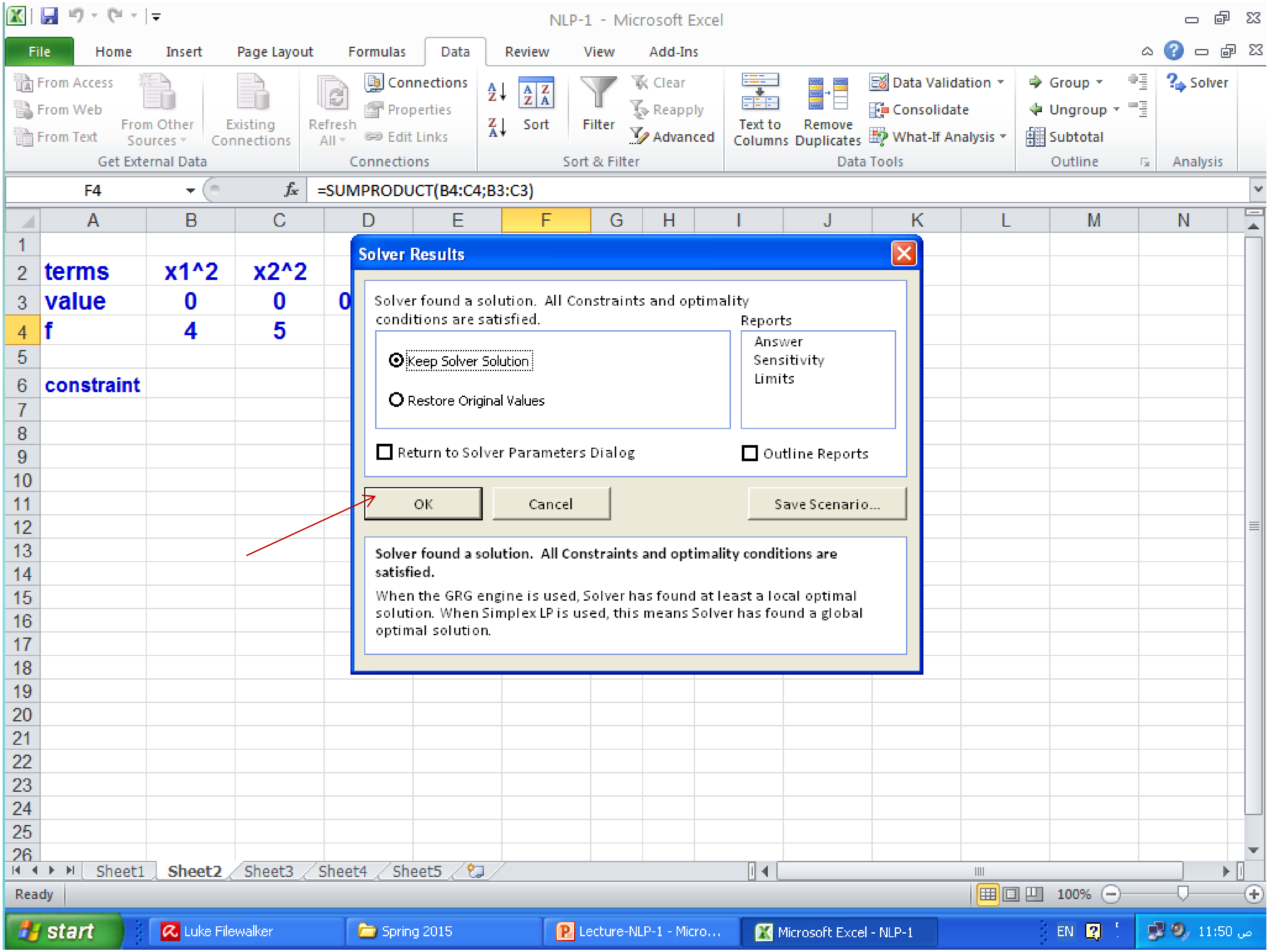
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Data Validation
Consolidate
What-If Analysis
Data Tools
Group
Ungroup
Subtotal
Outline
Solver
Analysis
F4
=SUMPRODUCT(B4:C4;B3:C3)
terms
x1^2
x2^2
value
0
0
f
4
5
constraint
Solver Results
Solver found a solution. All Constraints and optimality conditions are satisfied.
Reports
Answer
Sensitivity
Limits
Keep Solver Solution
Restore Original Values
Return to Solver Parameters Dialog
Outline Reports
OK
Cancel
Save Scenario...
Solver found a solution. All Constraints and optimality conditions are satisfied.
When the GRG engine is used, Solver has found at least a local optimal solution. When Simplex LP is used, this means Solver has found a global optimal solution.
Sheet1
Sheet2
Sheet3
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Sheet5
Ready
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Luke Filewalker
Spring 2015
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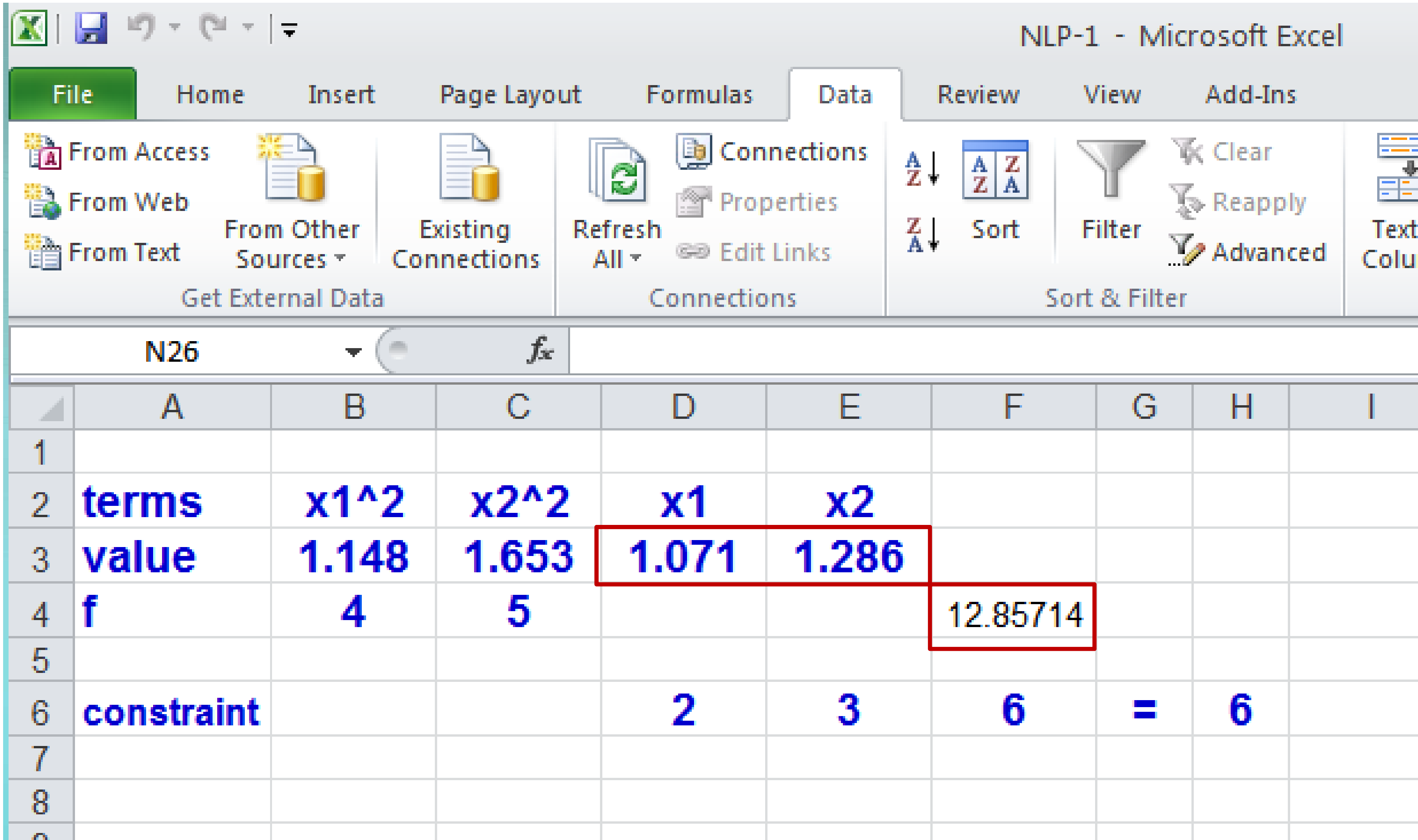


	A	B	C	D	E	F	G	H	I
1									
2	terms	x1^2	x2^2	x1	x2				
3	value	1.148	1.653	1.071	1.286				
4	f	4	5			12.85714			
5									
6	constraint			2	3	6	=	6	
7									
8									

Example 2

The profit analysis model:

Max the profit $z = v.p - c_f - v.c_v$ ………... (1)

The demand is represented by: $v = 1{,}500 - 24.6p$ ………... (2)

Where: v = volume (quantity), p = price,

c_f = fixed cost = \$10,000, c_v = variable cost = \$8 per unit.

Substituting values of c_f and c_v into (1), we obtain:

$z = v.p - 10{,}000 - 8v$ ………… (3)

Substituting (2) in (3):

$z = 1500p - 24.6p^2 - 10{,}000 - 8(1{,}500 - 24.6p)$

$\underline{z = 1696.8p - 24.6p^2 - 22{,}000}$ ………… (4)

$\frac{dz}{dp} = 1696.8p - 49.2p = 0$ for the critical points, then:

$p^* = 34.49$

$\frac{d^2z}{dp^2} = -49.2$ (<u>negative</u>), then p* is a <u>local maximum</u>.

Substituting in (2): $v^* = 1500 - 24.6(34.49) = 651.55$

Substituting in (3): $z_{max} = (651.55)(34.49) - 10{,}000 - 8(651.55) = 7259.56$

1.7.1.2 Lagrange Method

The basic features of the Lagrange multiplier method is given initially for a simple <u>problem of two variables with one constraint</u>.

The extension of the method to <u>a general problem of *n* variables with *m* constraints</u> is given later.

Problem with Two Variables and One Constraint.

Consider the problem Minimize $f(x_1,x_2)$

subject to: $g(x_1,x_2) = 0$

Define **Lagrange function** $L(x_1,x_2,\lambda) = f(x_1,x_2) + \lambda g(x_1,x_2)$

λ is called **the Lagrange multiplier.**

L is treated as a function of the three variables x_1, x_2, and λ.

Theorem: Necessary Conditions for Extremum:

$$\frac{\partial L}{\partial x_1}(x_1,x_2,\lambda) = \frac{\partial f}{\partial x_1}(x_1,x_2) + \lambda \frac{\partial g}{\partial x_1}(x_1,x_2) = 0$$

$$\frac{\partial L}{\partial x_2}(x_1,x_2,\lambda) = \frac{\partial f}{\partial x_2}(x_1,x_2) + \lambda \frac{\partial g}{\partial x_2}(x_1,x_2) = 0$$

$$\frac{\partial L}{\partial \lambda}(x_1,x_2,\lambda) = g(x_1,x_2) = 0$$

Theorem: Sufficient Condition

A sufficient condition for *f*(X) to have a relative minimum at X* is that the quadratic, *Q*, defined by:

$$Q = \frac{\partial^2 L}{\partial x_1 \partial x_2} dx_1 dx_2$$

evaluated at X = X* must be positive definite for all values of *dX* for which the constraints are satisfied.

- If Q is negative definite for all choices of the admissible variations *dX*, X* will be a constrained maximum of *f*(X).
- It has been shown by Hancock that a necessary condition for the quadratic form *Q*, to be positive (negative) definite for all admissible variations *d*X is that each root of the polynomial *Z*, defined by the following determinantal equation, be positive (negative):

$$\begin{vmatrix} L_{11} - z & L_{12} & g_{11} \\ L_{21} & L_{22} - z & g_{12} \\ g_{11} & g_{12} & 0 \end{vmatrix} = 0$$

where

$$L_{11} = \left.\frac{\partial^2 L}{\partial x_1^2}\right|_{(\mathbf{X}^*, \lambda^*)} \qquad L_{12} = \left.\frac{\partial^2 L}{\partial x_1\, \partial x_2}\right|_{(\mathbf{X}^*, \lambda^*)} = L_{21} \qquad L_{22} = \left.\frac{\partial^2 L}{\partial x_2^2}\right|_{(\mathbf{X}^*, \lambda^*)} = 0$$

$$g_{11} = \left.\frac{\partial g_1}{\partial x_1}\right|_{(\mathbf{X}^*, \lambda^*)} \qquad g_{12} = \left.\frac{\partial g_1}{\partial x_2}\right|_{(\mathbf{X}^*, \lambda^*)}$$

Example

Find the solution of the following problem using the Lagrange multiplier method:

$f(x,y) = x^{-1}y^{-2}$

Subject to: $g(x,y) = x^2 + y^2 - 4 = 0$

The Lagrange function is:

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^{-1}y^{-2} + \lambda(x^2 + y^2 - 4)$

The necessary conditions for the extreme of *f(x, y)* give:

$\frac{\partial L}{\partial x} = -x^{-2}y^{-2} + 2\lambda x = 0$ ……. (1) $\Rightarrow$ $\lambda = \frac{1}{2}x^{-3}y^{-2}$……… *(4)*

$\frac{\partial L}{\partial y} = -2x^{-1}y^{-3} + 2\lambda y = 0$ ……. (2) $\Rightarrow$ $\lambda = x^{-1}y^{-4}$ ……… *(5)*

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 4 = 0$ ……. (3)

From (4), (5): $\frac{1}{2}x^{-3}y^{-2} = x^{-1}y^{-4}$

$$\frac{1}{2}x^{-3}y^{-2} = x^{-1}y^{-4} \Rightarrow \frac{1}{2}y^{-2} = x^{2}y^{-4} \Rightarrow \frac{1}{2}y^{2} = x^{2}$$

$X^* = \frac{1}{\sqrt{2}}y^*$ **Or:** $x^2 = \frac{1}{2}y^2$ ………… (6)

(3): $x^2 + y^2 - 4 = 0$

From (6): $\frac{1}{2}y^2 + y^2 = 4 \Rightarrow y^2 = \frac{8}{3} \Rightarrow \boxed{y^* = 2\sqrt{\frac{2}{3}}}$

Substituting in (6): $\boxed{x^* = \frac{2}{\sqrt{3}}}$

(5): $\lambda = x^{-1}y^{-4} \Rightarrow \boxed{\lambda^* = \frac{\sqrt{3}}{2}\,\frac{1}{16}\,\frac{\sqrt{81}}{\sqrt{16}} = \frac{9\sqrt{3}}{128}}$

The determinant: $\begin{vmatrix} L_{11}-z & L_{12} & g_{11} \\ L_{21} & L_{22}-z & g_{12} \\ g_{11} & g_{12} & \mathbf{0} \end{vmatrix} = 0$

$$\begin{vmatrix} \left.\frac{\partial^2 L}{\partial x_1^2}\right|_{(\mathbf{X}^*,\lambda^*)} -z & \left.\frac{\partial^2 L}{\partial x_1\,\partial x_2}\right|_{(\mathbf{X}^*,\lambda^*)} & \left.\frac{\partial g_1}{\partial x_1}\right|_{(\mathbf{X}^*,\lambda^*)} \\ \left.\frac{\partial^2 L}{\partial x_1\,\partial x_2}\right|_{(\mathbf{X}^*,\lambda^*)} & \left.\frac{\partial^2 L}{\partial x_2^2}\right|_{(\mathbf{X}^*,\lambda^*)} -z & \left.\frac{\partial g_1}{\partial x_2}\right|_{(\mathbf{X}^*,\lambda^*)} \\ \left.\frac{\partial g_1}{\partial x_1}\right|_{(\mathbf{X}^*,\lambda^*)} & \left.\frac{\partial g_1}{\partial x_2}\right|_{(\mathbf{X}^*,\lambda^*)} & 0 \end{vmatrix} = 0$$

0

$$\begin{vmatrix} 1.218 - z & 0.344 & 2.309 \\ 0.344 & 0.974 - z & 3.266 \\ 2.309 & 3.266 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1.218-z & 0.344 & 2.309 \\ 0.344 & 0.974-z & 3.266 \\ 2.309 & 3.266 & 0 \end{vmatrix} =$$

$$(1.218-z)\begin{vmatrix} 0.974-z & 3.266 \\ 3.266 & 0 \end{vmatrix} - (0.344)\begin{vmatrix} 0.344 & 3.266 \\ 2.309 & 0 \end{vmatrix} + 2.309\begin{vmatrix} 0.344 & 0.974-z \\ 2.309 & 3.266 \end{vmatrix}$$

$$(1.218-z)\left[(0.974-z)(0) - (3.266)(3.266)\right] - (0.344)\left[-(3.266)(2.309)\right] +$$

$$+ (2.309)\left[(0.344)(3.266) - (0.974-z)(2.309)\right] = 0$$

Z= - 0.344 (negative), Then x*,y* is a relative maximum

*f**(x*,y*) = 0.3248

Determinant Calculation

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} \\ &= -3 + 24 - 24 \end{aligned}$$

$$= -3$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix}.$$

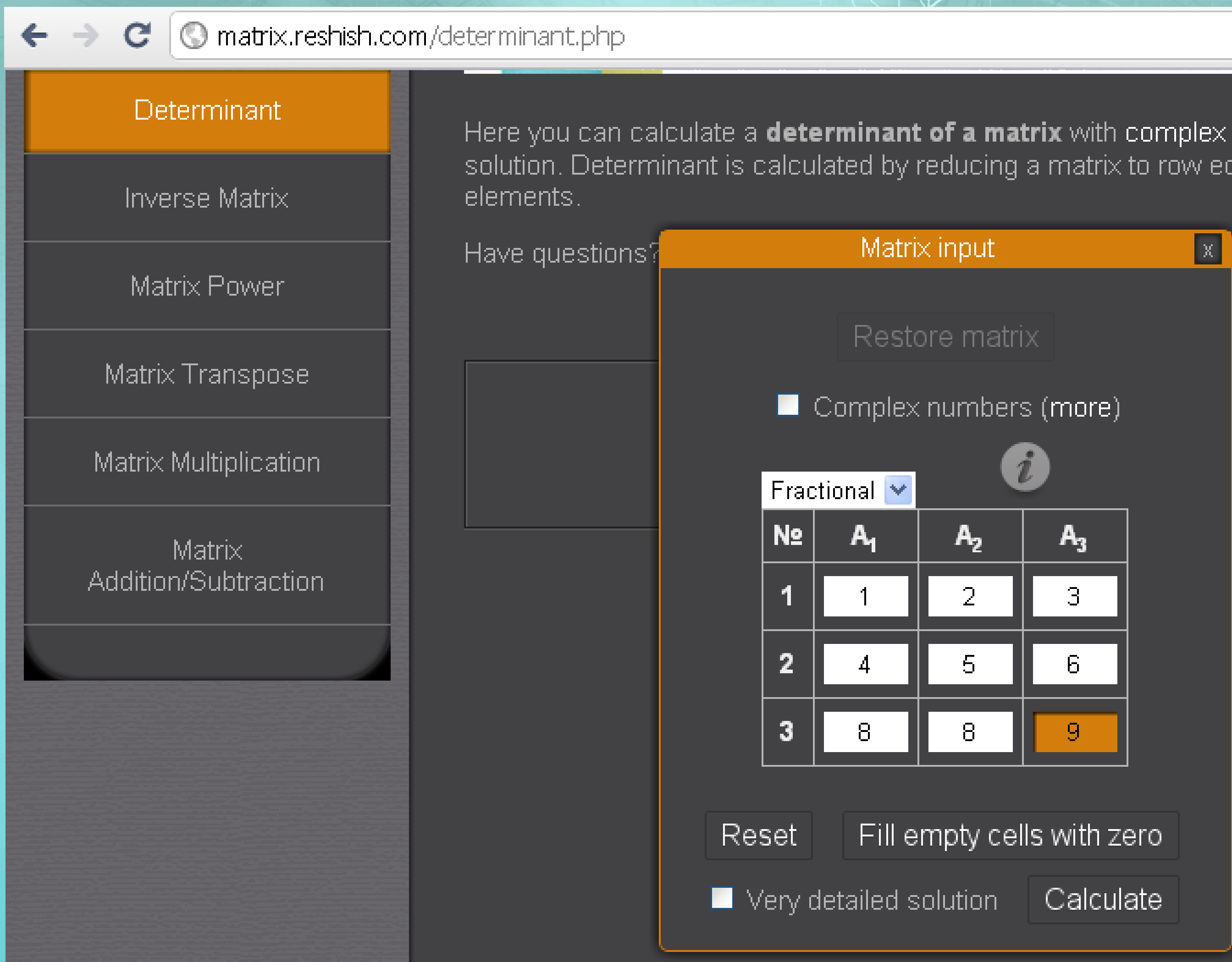
Solution. Using row operations $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 8R_1$ and then expanding along the first column, gives

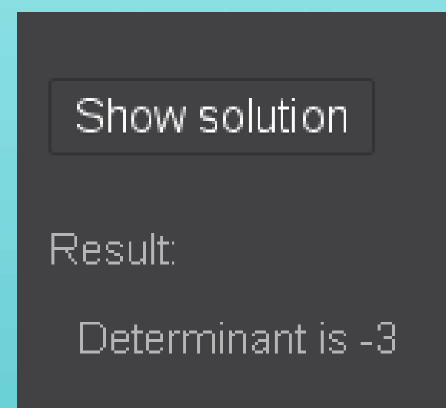
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -8 & -15 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -8 & -15 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 \\ -8 & -15 \end{vmatrix} = \qquad R_2 \longrightarrow R_2 + 8R_1$$

$$= -3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -3[1(1)-2(0)] = -3(1) = -3$$

http://matrix.reshish.com/determinant.php





Necessary Conditions for a General Problem

The equations can be extended to the case of a general problem with *n* variables and *m* equality constraints:

Minimize $f(\mathbf{X}) = f(x_1, x_2, \ldots\ldots, x_n)$

subject to: $g_j(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, m$

The Lagrange function, L, in this case is defined by introducing one Lagrange multiplier λ_j for each constraint $g_j(X)$ as:

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(\mathbf{X}) + \lambda_1 g_1(\mathbf{X}) + \lambda_2 g_2(\mathbf{X}) + \cdots + \lambda_m g_m(\mathbf{X})$$

The necessary conditions for the extremum of L, are given by:

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, m$$

Sufficient Condition

A sufficient condition for *f*(X) to have relative minimum at X* is that the quadratic, *Q*, defined by:

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \, \partial x_j} dx_i \, dx_j$$

evaluated at X = X* must be positive definite for all values of variations *dX* for which the constraints are satisfied.

If Q is negative definite for all choices of the admissible variations dx_i, X* will be a constrained maximum of *f*(X).

It has been shown by Hancock that a necessary condition for the quadratic form *Q*, to be positive (negative) definite for all admissible variations dX is that each root of the polynomial *Zi*, defined by the following determinantal equation, be positive (negative):

$$\begin{vmatrix} L_{11} - z & L_{12} & L_{13} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\ L_{21} & L_{22} - z & L_{23} & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\ \vdots & & & & & & & & \\ L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} - z & g_{1n} & g_{2n} & \cdots & g_{mn} \\ g_{11} & g_{12} & g_{13} & \cdots & g_{1n} & 0 & 0 & \cdots & 0 \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ g_{m1} & g_{m2} & g_{m3} & \cdots & g_{mn} & 0 & 0 & \cdots & 0 \end{vmatrix} = 0$$

Where:

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \, \partial x_j}(\mathbf{X}^*, \lambda^*), \quad g_{ij} = \frac{\partial g_i}{\partial x_j}(\mathbf{X}^*)$$

This equation on expansion, leads to an *(n* — m)th-order polynomial in *z.* If some of the roots of this polynomial are positive while the others are negative, the point X* is not an extreme point.

1.7.2 Multivariable Optimization With Inequality Constraints

Consider the following problem: Minimize $f(X)$ subject to: $g_j(X) \leq 0$, $j = 1,2,...,m$

Kuhn-Tucker Conditions

The conditions to be satisfied at a constrained minimum point, X*. These conditions are, in general, not sufficient to ensure a relative minimum. However, there is a class of problems, called *convex programming problems* for which the Kuhn-Tucker conditions are necessary and sufficient for a global minimum.

Kuhn-Tucker Conditions

The Kuhn-Tucker conditions can be stated as follows:

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i = 1,2,\ldots,n$$

$$\begin{aligned} \lambda_j g_j &= 0, \qquad & j &= 1,2,\ldots,m \\ g_j &\leq 0, \qquad & j &= 1,2,\ldots,m \\ \lambda_j &\geq 0, \qquad & j &= 1,2,\ldots,m \end{aligned}$$

Note that if the problem is one of maximization or if the constraints are of the type $g_j \geq 0$, the λ_j have to be nonpositive. On the other hand, if the problem is one of maximization with constraints in the form $g_j \geq 0$, the λ_j have to be nonnegative.

Types of Nonlinear Programming

- **Nonlinear objective function, linear constraints.**
- **Nonlinear objective function and nonlinear constraints.**
- **Linear objective function and nonlinear constraints.**

Nonlinear Objective Function and Linear Constraints:

The Great Western Appliance Company sells two models of toaster ovens, the Micro toaster (X_1) and the Self-Clean Toaster Oven (X_2).
The firm earns a profit of $28 for each Micro toaster regardless of the number sold. Profits for the Self-Clean model, however, increase as more units are sold because of fixed overhead. Profit on this model may be expressed as 21 X_2 + 0.25 X_2^2.
Great Western's profit is subject to two linear constraints on production capacity and sales unit time available.

Max: $28X_1 + 21X_2 + 0.25X_2^2$

Subject to:

$$X_1 + X_2 \le 1000 \text{ (units of production capacity)},$$
$$0.5X_1 + 0.4X_2 \le 500 \quad \text{(hours of sales time available)}.$$
$$X_1, X_2 \ge 0.$$

When an objective function contains a squared term and the problem constraints are linear, it is called a ***quadratic programming*** problem.

An Excel Formulation of Great Western Appliance's Nonlinear Programming Problem.

PROGRAM 10.9
Excel 2010 Solver Solution for Great Western Appliance NLP Problem

	A	B	C	D	E	F	G
1	Great Western Appliance						
2		Micro	Self-Clean				
3	Variables	X1	X2				
4	Values	0	1000				
5							
6	Terms	X1	X2	$X2^2$			
7	Calculated Values	0	1000	1000000	Profit		
8	Profit	28	21	0.25	21000		
9							
10	Constraints				LHS	Sign	RHS
11	Capacity	1	1		1000	≤	1000
12	Hours Available	0.5	0.4		400	≤	500

Key Formulas

	E
8	=SUMPRODUCT($B$7:$D$7,B8:D8)
9	
10	LHS
11	=SUMPRODUCT($B$4:$C$4,B11:C11)
12	=SUMPRODUCT($B$4:$C$4,B12:C12)

	B	C	D
7	=B4	=C4	=C4^2

Solver Parameter Inputs and Selections

Set Objective: E8

By Changing cells: B4:C4

To: Max

Subject to the Constraints:

E11:E12 <= G11:G12

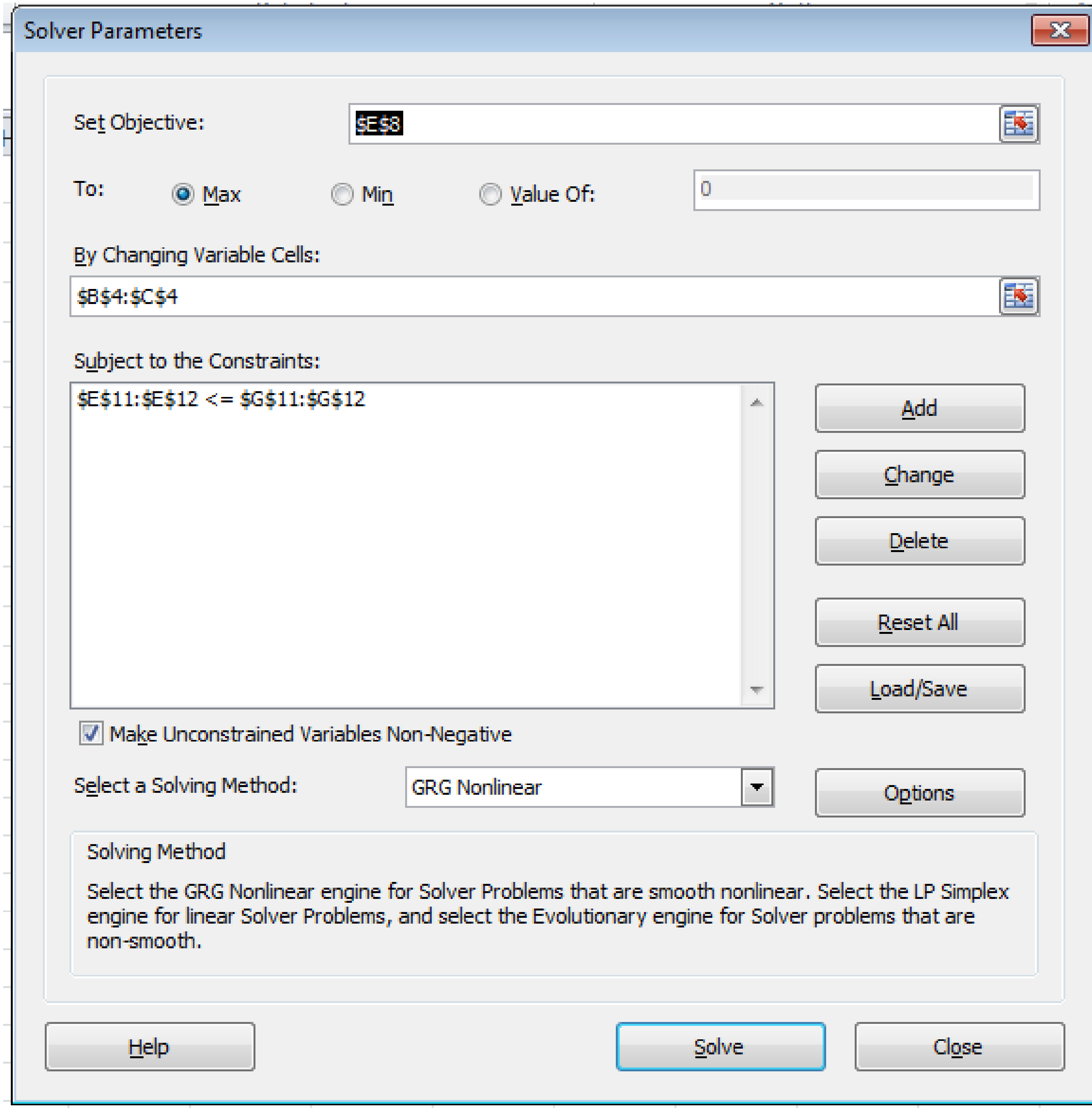
Solving Method: GRG Nonlinear

☑ Make Variables Non-Negative

Input Screen

$$\text{Max: } 28X_1 + 21X_2 + 0.25X_2^2$$
$$X_1 + X_2 \le 1000$$
$$0.5X_1 + 0.4X_2 \le 500$$

	A	B	C	D	E	F	G	H
1	Great Western Applience							
2		Micro	Self-Clean					
3	Variables	X1	X2					
4	values							
5								
6	Terms	X1	X2	$X2^2$				
7	Calculated values	0	0	0	Profit			
8	Profit	28	21	0.25	0			
9								
10	Constraints				LHS	Sign	RHS	
11	Capacity	1	1		0	≤	1000	
12	Hours available	0.5	0.4		0	≤	500	
13								


Solver Parameters
Set Objective:
$E$8
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$B$4:$C$4
Subject to the Constraints:
$E$11:$E$12 <= $G$11:$G$12
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
GRG Nonlinear
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
Close

Solution to Great Western Appliance's NLP Problem using Excel Solver:

	A	B	C	D	E	F	G	H
1	Great Western Applience							
2		Micro	Self-Clean					
3	Variables	X1	X2					
4	values	0	1000					
5								
6	Terms	X1	X2	$X2^2$				
7	Calculated values	0	1000	1000000	Profit			
8	Profit	28	21	0.25	271000			
9								
10	Constraints				LHS	Sign	RHS	
11	Capacity	1	1		1000	≤	1000	
12	Hours available	0.5	0.4		400	≤	500	
13								

Both Nonlinear Objective Function and Nonlinear Constraints.

The annual profit at a medium-sized (200-400 beds) Hospital Corporation-owned hospital depends on the number of medical patients admitted (X_1) and the number of surgical patients admitted (X_2). The nonlinear objective function for Hospicare is:

Max. $13X_1 + 6X_1X_2 + 5X_2 + 1/X_2$.

The corporation identifies three constraints, two of which are also nonlinear, that affect operations. They are

$2X_1^2 + 4X_2^2 \leq 90$ (nursing capacity, in thousands of labor-days).

$X_1 + \qquad \leq 75$ (x-ray capacity, in thousands).

$8X_1 - X_2^3 2X_2 \leq 61$ (marketing budget required, in thousands of $).

Max: $13X_1 + 6X_1X_2 + 5X_2 + 1/X_2$

Subject to:

$$2X_1^2 + 4X_2^2 \le 90$$
$$X_1 + X_2^3 \le 75$$
$$8X_1 - 2X_2 \le 61$$

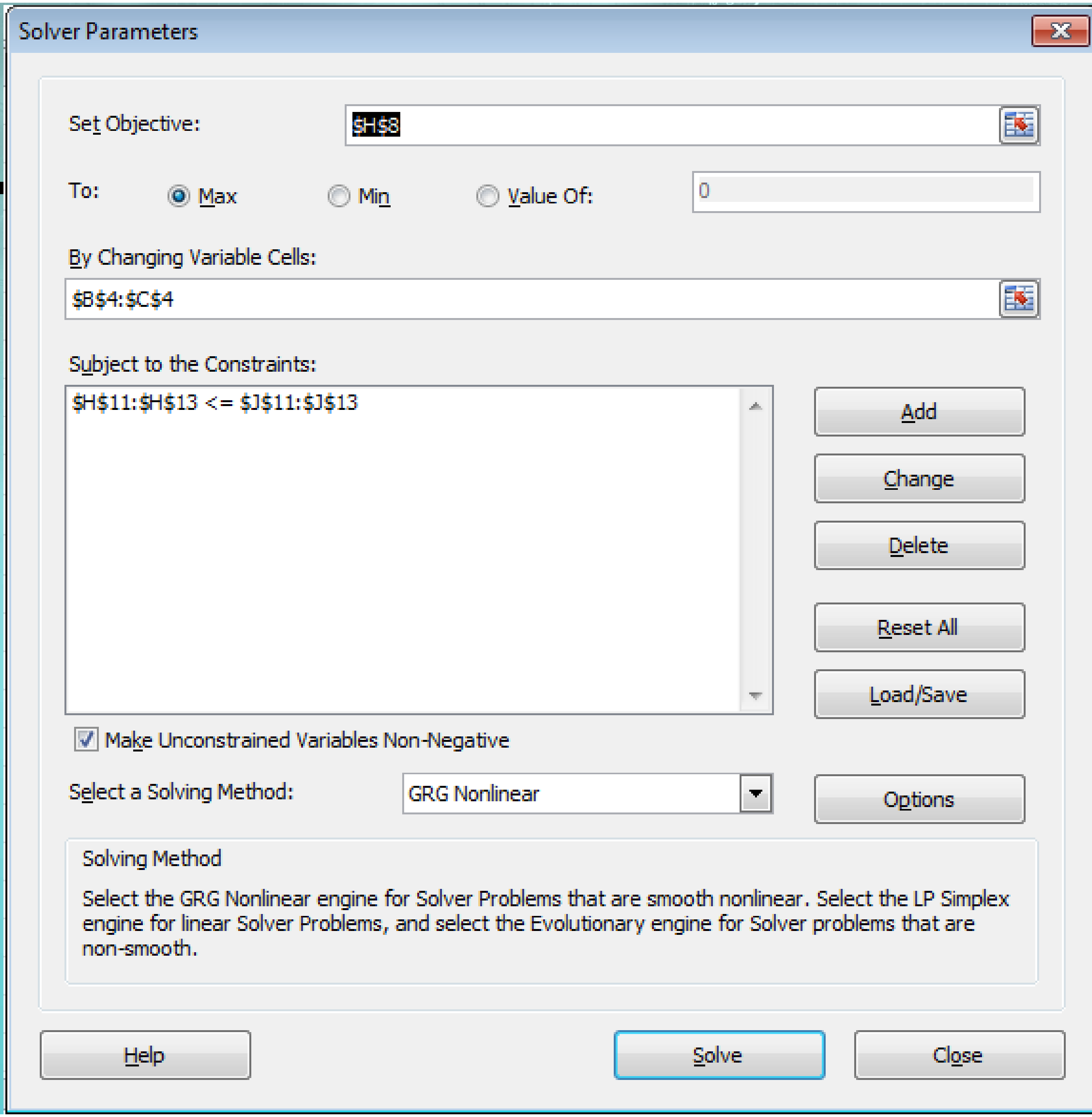
An Excel Formulation of Hospicare Corp.'s NLP Problem:

R25 | f_x

	A	B	C	D	E	F	G	H	I	J
1	Hospicare Corp									
2										
3	Variables	X1	X2							
4	Values	1	1							
5										
6	Terms	X1	$X1^2$	X1*X2	X2	$X2^3$	1/X2			
7	Calculated Values	1	1	1	1	1	1	Total Profit		
8	Profit	13	0	6	5		1			
9										
10	Constraints							LHS	Sign	RHS
11	Nursing		2		4				≤	90
12	X-Ray	1				1			≤	75
13	Budget	8			-2				≤	61
14										
15										
16										

Max: $13X_1 + 6X_1X_2 + 5X_2 + 1/X_2$

$$2X_1^2 + 4X_2^2 \le 90$$
$$X_1 + X_2^3 \le 75$$
$$8X_1 - 2X_2 \le 61$$


Solver Parameters
Set Objective:
$H$8
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$B$4:$C$4
Subject to the Constraints:
$H$11:$H$13 <= $J$11:$J$13
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
GRG Nonlinear
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
Close

	A	B	C	D	E	F	G	H	I	J
2										
3	Variables	X1	X2							
4	Values	6.066259	4.1							
5										
6	Terms	X1	$X1^2$	X1*X2	X2	$X2^3$	1/X2			
7	Calculated Values	6.066259	36.79949	24.87319	4.100253	68.93374	0.244	Total Profit		
8	Profit	13	0	6	5		1	248.845671		
9										
10	Constraints							LHS	Sign	RHS
11	Nursing		2		4			89.999998	≤	90
12	X-Ray	1				1		75	≤	75
13	Budget	8			-2			40.329564	≤	61
14										

Linear Objective Function with Nonlinear Constraints

Thermlock Corp. produces massive rubber washers and gaskets like the type used to seal joints on the NASA Space Shuttles. To do so, it combines two ingredients; rubber (X_1) and oil (X_2).
The cost of the industrial quality rubber used is $5 per pound and the cost of the high viscosity oil is $7 per pound. Two of the three constraints Thermlock faces are nonlinear. The firm's objective function and constraints are

Min. $5X_1 + 7X_2$

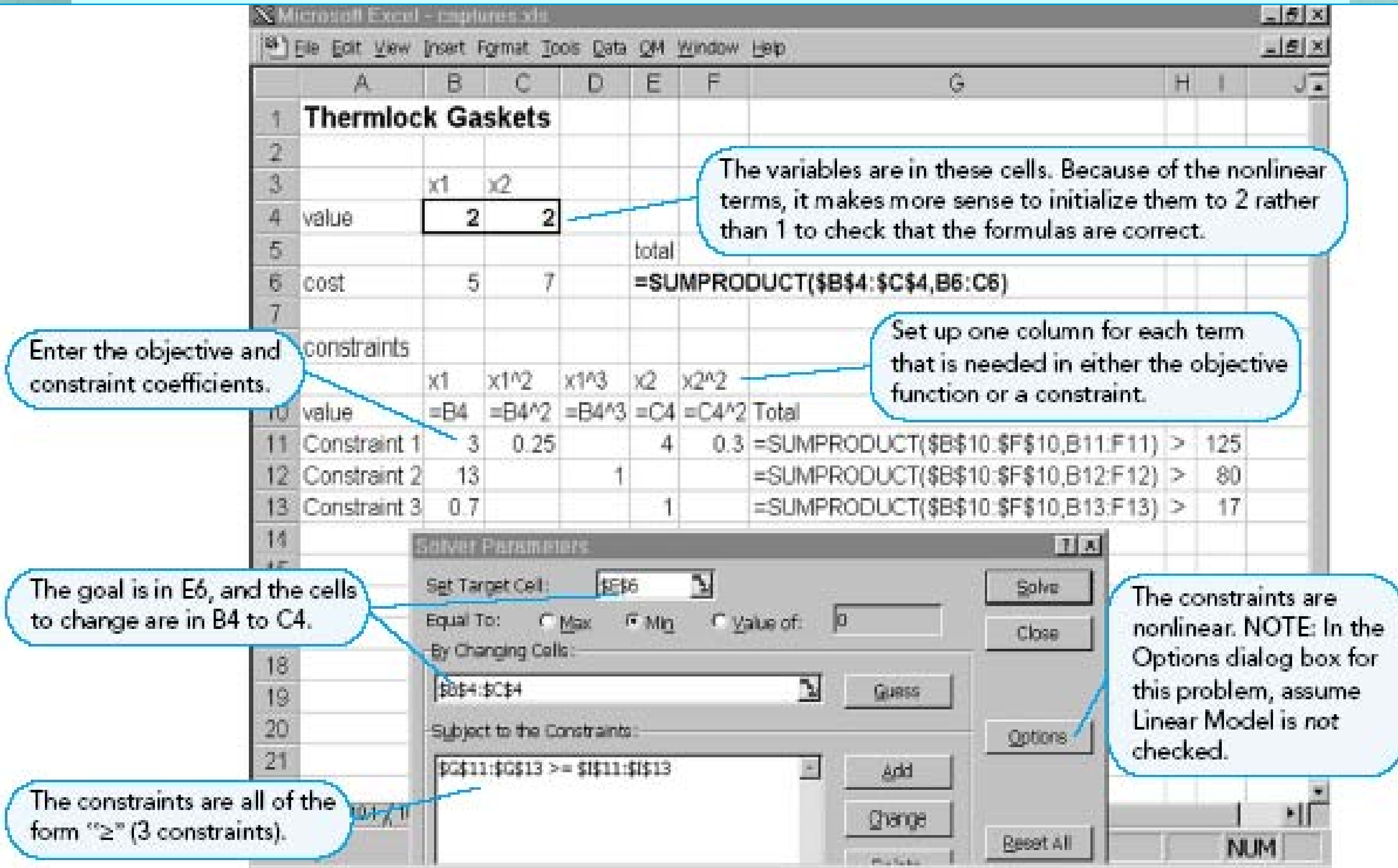
Subject to:

$3X_1 + 0.25X_1^2 + 4X_2 + 0.3X_2^2 \geq 125$ (hardness constraint),

$13X_1 + X_1^3 \geq 80$ (tensile strength),

$0.7X_1 + X_2 \geq 17$ (elasticity).

Excel Formulation of Thermlock's NLP Problem:



Solution to Thermlock's NLP Problem Using Excel Solver:

Microsoft Excel - captures.xls

File Edit View Insert Format Tools Data QM Window Help

	A	B	C	D	E	F	G	H	I
1	**Thermlock Gaskets**								
2									
3		x1	x2						
4	value	**3.325326**	**14.67227**						
5					total				
6	cost	5	7		**119.3325**				
7									
8	constraints								
9		x1	x1^2	x1^3	x2	x2^2			
10	value	3.325326	11.05779	36.77076	14.67227	215.2756	Total		
11	Constraint 1	3	0.25		4	0.3	136.0122	>	125
12	Constraint 2	13		1			80	>	80
13	Constraint 3	0.7			1		17	>	17

Computational Procedures -Nonlinear Programming

Unlike LP methods:

- **One disadvantage of NLP is that the solution procedures to solve nonlinear problems do not always yield an optimal solution in a finite number of steps. The solution yielded may only be a local optimum, rather than a global optimum. In other words, it may be an optimum over a particular range, but not overall.**
- **There is no general method for solving all nonlinear problems.**
- **Classical optimization techniques based on calculus, can handle some special cases, usually simpler types of problems.**

Gradient method (steepest ascent method)

It is an interactive procedure that moves from one feasible solution to the next in improving the value of the objective function.

It has been computerized and can handle problems with both nonlinear constraints and objective.

Separable programming

- **Linear representation of nonlinear problem.**
- ***Separable programming*** **deals with a class of problems in which the objective and constraints are approximated by linear functions. In this way, the powerful simplex algorithm may again be applied.**

In general, work in the area of NLP is the least charted and the most difficult of all the quantitative analysis models.